A few basic structures determine the behavior of a coupled map lattice

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The stability and formation of structures in lattices of diffusively coupled logistic maps are investigated for high nonlinearity and medium and large coupling. Two stability statements are given that relate the presence of the predominant attractors, i.e., cycles and quasiperiodic traveling waves, to the stability of a few simple periodic structures. They are supported by strong numerical evidence. Furthermore, they are justified through the description of some mechanisms that connect the formation of a stable structure to the cycles of the uncoupled lattice. As an important consequence, for given parameter values, an approximate prediction of the behavior of the lattice is allowed. [S1063-651X(98)15402-8]

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I. INTRODUCTION

Coupled map lattices have recently been introduced as simple models for the study of the dynamics of spatially extended systems with (infinitely) many degrees of freedom. Such models, in spite of their simple form, can exhibit a wide variety of complex behaviors. For this reason, they can represent an adequate tool to investigate the spatiotemporal phenomena which occur in the real world. The subject is amply covered in the literature [1–4].

Among others, a key open problem is to determine the mechanisms by which spatiotemporal structures are selected and then acquire physical significance. The question can be reformulated by asking which patterns can acquire stability and why those and not others. In the present paper we offer a contribution that we trust will help to tackle the problem.

We consider logistic maps with diffusive coupling, namely, the lattice map $\Phi^N_{\epsilon,a}:(-1,1)^N \rightarrow (-1,1)^N$, defined by

$$x_i^{k+1} = (1-\epsilon) f(x_i^k) + \frac{\epsilon}{2} [f(x_{i-1}^k) + f(x_{i+1}^k)], \qquad (1)$$

with periodic boundary conditions. k is the discrete time and $\epsilon \in [0,1]$ is the coupling parameter. The logistic map $f(x)$, which provides the local evolution, is expressed in the form

$$f(x) = 1 - ax^2, \qquad (2)$$

with the nonlinearity parameter a varying in $(0,2]$.

We investigate lattice (1) for medium and large coupling and high nonlinearity, say $\epsilon \geqslant 0.4$ and $a \geqslant 1.6$. In this parameter region the asymptotic dynamics, often generically referred to as pattern selection with suppression of chaos, is mainly due to cycles and quasiperiodic traveling waves of period 4. These attractors are largely predominant for large N, say $N \geqslant 100$. For smaller N cycles of period 2 are also quite relevant to the dynamics. We, therefore, take into account cycles of periods 2 and 4 and traveling waves of period 4.

Very recently an interesting numerical result [5] was obtained which concerns the structures of time period 2 for small coupling ($\epsilon < 0.4$). That is, it was shown that the stability of a structure is determined by the stability of its finest patterns. These patterns belong to a set of a few ''fundamental cycles'' with the property that any other cycle can be seen as being composed of some of them. Each composed cycle is stable if all the component cycles are stable, and its stability region approximately corresponds to the intersection of the stability regions of all the components.

In accordance with Ref. [5], we follow this simple idea: a spatial structure (wave) can be seen as the composition of D elementary structures whose stability somehow determines that of the whole structure. We will show that the occurrence of cycles and quasiperiodic traveling waves can be associated with the stability of a few basic cycles of length m between 5 and 12, which represent the only wavelengths that are allowed in the parameter region considered. The association is formalized by two stability statements: one for period 2, the other for period 4. With our knowledge of the stability regions of the basic cycles, the two statements have a remarkable implication: the possibility of predicting most of the nonchaotic behavior of lattice (1).

To predict in what sense? Given a lattice of size N, let us consider any spatial structure (a cycle of period 2 or 4, or a quasiperiodic traveling wave of period 4) composed of D elementary waves. If it is stable, its stability region will be included in a larger region which is determined by the pair (N,D). It is this latter region that can be more or less accurately predicted. Note that, substantially, it is the ''basic'' wavelength $[N/D]$ that, independently of N, determines the parameter region where a structure may be stable.

In a sense, the stability statements given here extend an analogous statement given in Ref. [5] to a wider parameter region and to a wider class of structures. There are, however, some differences that appear worthy of note. In Ref. [5] the result was quite precise, but relevant only to cycles of period 2. Unfortunately, these cycles govern only a small percentage of the dynamics that occurs for $\epsilon < 0.4$, and certainly not the most interesting part. On the other hand, the results obtained here, though rougher, concern most of the dynamics of major interest for $\epsilon \geqslant 0.4$. In our opinion, such greater generality, with the simultaneous treatment of cycles and quasiperiodic traveling waves, constitutes the interesting aspect of this paper.

II. PRELIMINARY REMARKS AND NOTATIONS

Numerical investigation shows that in the parameter region under consideration, nonchaotic attractors, i.e., cycles and tori, are clearly predominant. Most periodic behavior is determined by p-periodic cycles with a particular characteristic: they are nodes of a *heteroclinic cycle* [4]. This is an invariant one-dimensional manifold formed by p (or $p/2$) curves and including two distinct families of N cycles of period p, the ones being stable (nodes), the others being unstable (saddles). The manifold is the union of the $2pN$ points of the cycles with the unstable manifolds of the saddles.

A heteroclinic cycle can also be regarded as a phase-locked torus with rotation number $w=0$ (or $w=1/2$). Such a torus has the special property that nodes and saddles have an eigenvalue of their stability matrix which remains very close to one as the parameters are varied. For the nodes this implies attractivity with relaxation times much longer, sometimes very much longer, than those of a normal cycle. The knowledge of the existence of cycles with this peculiarity is of fundamental importance for the definition of the overall behavior of Eq. (1). However, for the sake of simplicity, we will not distinguish between normal cycles and cycles located on a heteroclinic cycle [6].

Most tori are two-tori, and most of them are *quasiperiodic traveling waves* [2]. These are characterized by a very small rotation number (in our experiments of the order of 10^{-4} or smaller). Consequently, the associated pattern moves quite slowly in space. Henceforth, quasiperiodic traveling waves will be simply referred to as traveling waves.

A two-torus consists of p distinct closed curves which are invariant under $(\Phi^N_{\epsilon,a})^p$. For this reason one can say that a two-torus, like a cycle, has a period p. In all our experiments we found $p=2^k$, with $k\geqslant 1$ in the case of a cycle and $k\geqslant 2$ in the case of a traveling wave.

A trajectory point of Eq. (1) can be seen as a spatial structure (or wave) that may evolve with time. For the attractors we are interested in, that is, cycles and quasiperiodic traveling waves, the spatial structure retains its form. Clearly, whereas for a cycle it is fixed under $(\Phi^N_{\epsilon,a})^p$, for a traveling wave it translates very slowly, recovering its initial position, though not exactly, after p/w iterations.

A spatial structure is significantly characterized by two numbers: its size N and its number of domains D. The latter can be defined as the number of times the spatial wave crosses the line $x=z^0_0$ from below, z^0_0 being the fixed point of the logistic map. D can also be regarded as the number of elementary waves that compose the structure (see Fig. 1).

Here we consider only attractors of periods 2 and 4, which govern most of the dynamics of lattice (1). Note that, as size N becomes larger and larger, the relevance of period 2 gradually decreases until it vanishes. Hence, for large N, period 4 is largely predominant.

Most of our results were obtained by means of a computational code, mainly based on the calculation of the Liapunov exponents, which is able to define the nature of the attractor onto which a trajectory relaxes. In particular, this code distinguishes between normal cycles and cycles belonging to a heteroclinic cycle, between two- and three-tori and between normal tori and traveling waves. Furthermore, when the attractor is a cycle or a traveling wave, it determines its period p and its number of domains D.

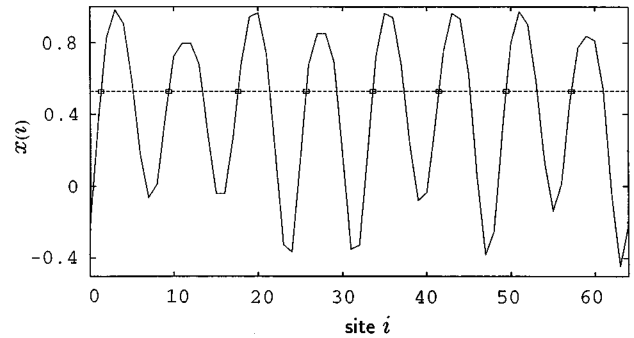

FIG. 1. Spatial structure associated with a traveling wave for $N=64$, $a=1.7$, and $\epsilon=0.55$. In this case the number D of domains is 8.

It should be pointed out that a possible reason, perhaps the only one, why an attractor may be incorrectly classified is due to the well-known phenomenon of supertransients. Although our computations take into account the possible occurrence of such a phenomenon, we cannot exclude the possibility of a nonchaotic attractor (most likely a cycle) being erroneously classified as chaotic. However, we can state that attractors are never erroneously classified as cycles or traveling waves. This is of fundamental importance for the reliability of our results.

Some notations are now needed. Let the nonlinearity parameter a be fixed, and equal to $\overline{a}$. We denote by $z^k_i=f(z^k_{i-1})$, $i=0,...,2^k-1$, the points of the cycle of the logistic map of period 2^k originating from the fixed point z^0_0. As is well known, all these cycles are unstable for the values of a being taken into account. For the sake of argument, let us suppose that $z^k_0=\min_i z^k_i$.

A lattice point of the form

$$(z^{k_1}_{c_1},z^{k_2}_{c_2},\dots,z^{k_N}_{c_N}),\quad k_j\in\{0,1,...\},\quad c_j\in\{0,...,2^{k_j}-1\} \quad (3)$$

is a point of an unstable cycle of period $p=2^k$, $k=\max_{1\leqslant j\leqslant N}k_j$, for the uncoupled map $\Phi^N_{0,\overline{a}}$.

We will be interested, in particular, in cycles of period 2^k in which all the k_j's are equal to k, except at most a few that are equal to 0. This allows us to use the simpler notation

$$(c_1c_2\dots c_N)_k, \quad (4)$$

with $c_j\in\{\overline{0},0,...,2^k-1\}$, and $\overline{0}$ denoting the fixed point z^0_0. Notation (4) can be further simplified in the presence of i consecutive c_j's equal to each other by representing such a subsequence by c^i_j. So, for instance, $(0^4\overline{0}1^2)_1$ corresponds to $(0000\overline{0}11)_1$ and $(0^43^4)_2$ to $(00003333)_2$.

As ϵ is increased from zero, every cycle of form (3) is the origin of a one-dimensional continuum of cycles of $\Phi^N_{\epsilon,\overline{a}}$ that retain the same spatial structure. For this reason, we continue to refer to it by the same notation as for the originating cycle. An efficient numerical tool for the investigation of the continuation of a cycle as ϵ varies is the Newton method. We use it mainly to find out whether or not a cycle acquires stability, and when and how it comes to an end.

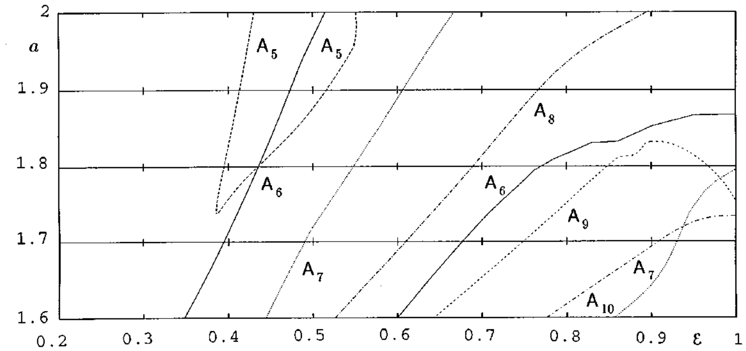

FIG. 2. Stability regions $\mathcal{R}(\mathbf{A}_m)$, $m=5,...,10$. Continuous lines mark the boundary of $\mathcal{R}(\mathbf{A}_6)$, broken lines that of $\mathcal{R}(\mathbf{A}_5)$ and $\mathcal{R}(\mathbf{A}_9)$, dotted lines that of $\mathcal{R}(\mathbf{A}_7)$, and dotted-broken lines that of $\mathcal{R}(\mathbf{A}_8)$ and $\mathcal{R}(\mathbf{A}_{10})$. The symbols A_m are placed inside the associated region near the boundary curves.

III. BASIC CYCLES

As stated in Sec. I, we set out to show that a nonchaotic structure of any size N, be it a cycle or a traveling wave, can be regarded as the composition of D elementary waves whose stability determines that of the whole structure. The cycles of the form $(0^i1^{m-i})_k$ are the simplest structures of period 2^k, present for $\epsilon=0$, that one can associate with an elementary wave of size m. For this reason, we first determined the stability of the cycles of periods 2 and 4 of this type. We found that only a few of them, all of size $5\leqslant m\leqslant 12$, succeed in acquiring stability in the parameter region under consideration. In addition, in the case of period 2, we found that some cycles with one or two coordinates equal to z_0^0 also become stable.

The cycles that become stable are listed below grouped in sets $\mathbf{A}_m$ (period 2) and $\mathbf{B}_m$ (period 4) according to their size:

$$\mathbf{A}_5\equiv\{(0^31^2)_1\},$$

$$\mathbf{A}_6\equiv\{(0^41^2)_1,(0^31^3)_1,(\overline{00}^2\overline{01}^2)_1\},$$

$$\mathbf{A}_7\equiv\{(0^41^3)_1,(0^3\overline{01}^3)_1\},$$

$$\mathbf{A}_8\equiv\{(0^41^4)_1,(0^51^3)_1\},$$

$$\mathbf{A}_9\equiv\{(0^61^3)_1,(0^51^4)_1,(0^4\overline{01}^4)_1\},$$

$$\mathbf{A}_{10}\equiv\{(0^61^4)_1,(\overline{00}^4\overline{01}^4)_1\},$$

$$\mathbf{B}_5\equiv\{(0^31^2)_2\},$$

$$\mathbf{B}_6\equiv\{(0^41^2)_2,(0^31^3)_2\},$$

$$\mathbf{B}_7\equiv\{(0^31^4)_2,(0^41^3)_2\},$$

$$\mathbf{B}_8\equiv\{(0^41^4)_2,(0^51^3)_2\},$$

$$\mathbf{B}_9\equiv\{(0^41^5)_2,(0^51^4)_2\},$$

$$\mathbf{B}_{10}\equiv\{(0^61^4)_2,(0^71^3)_2\},$$

$$\mathbf{B}_{11}\equiv\{(0^31^8)_2,(0^41^7)_2,(0^71^4)_2,(0^81^3)_2\},$$

$$\mathbf{B}_{12}\equiv\{(0^31^9)_2,(0^41^8)_2,(0^81^4)_2,(0^91^3)_2\}.$$

Henceforth, a cycle belonging to $\mathbf{A}_m$ ($\mathbf{B}_m$) will be generically denoted by $\mathcal{A}_m$ ($\mathcal{B}_m$). Furthermore, let $\mathcal{R}(\mathbf{A}_m)$ [$\mathcal{R}(\mathbf{B}_m)$] represent the region, in the parameter plane (ϵ,a), where a stable cycle of the map $\Phi^m_{\epsilon,a}$ exists, which is the continuation of a cycle $\mathcal{A}_m$ ($\mathcal{B}_m$). For each m and any given point (ϵ,a) belonging to $\mathcal{R}(\mathbf{A}_m)$ [$\mathcal{R}(\mathbf{B}_m)$], just one $\mathcal{A}_m$ ($\mathcal{B}_m$) is stable.

The stability regions $\mathcal{R}(\mathbf{A}_m)$ and $\mathcal{R}(\mathbf{B}_m)$ are shown in Figs. 2 and 3. It should be noted that, also in the case where $\mathbf{A}_m$ ($\mathbf{B}_m$) consists of more than one cycle, $\mathcal{R}(\mathbf{A}_m)$ [$\mathcal{R}(\mathbf{B}_m)$] is a connected set. This is because, in most cases, a stable cycle $\mathcal{A}_m$ ($\mathcal{B}_m$) is involved in a heteroclinic cycle [4] together with another unstable $\mathcal{A}_m$ ($\mathcal{B}_m$), and a typical bifurcation phenomenon of a heteroclinic cycle is the exchange of stability between the cycles lying on it.

In what follows, the interval of ϵ that corresponds to the intersection of the stability region $\mathcal{R}(\mathbf{A}_m)$ with the line $a=\overline{a}$ will be denoted by (α_m^1,α_m^2), with the assumption $\alpha_m^2=1$ in the case where $\mathcal{R}(\mathbf{A}_m)$ extends to $\epsilon=1$. Analogously, (β_m^1,β_m^2) will represent the intersection of the same line with $\mathcal{R}(\mathbf{B}_m)$. From Fig. 2 it is clear that, for any $\overline{a}$, we have

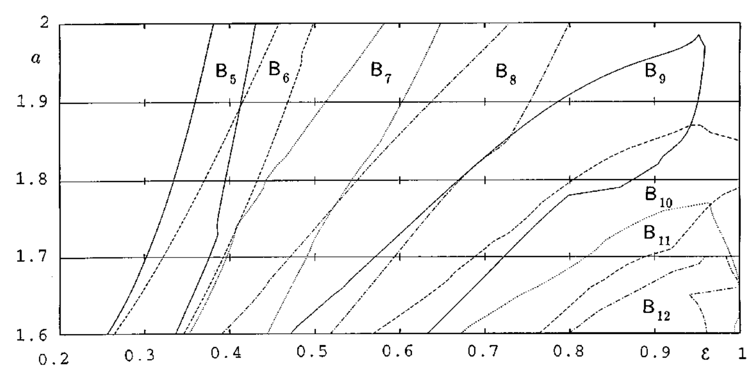

FIG. 3. Stability regions $\mathcal{R}(\mathbf{B}_m)$, $m=5,...,12$. Continuous lines mark the boundary of $\mathcal{R}(\mathbf{B}_5)$ and $\mathcal{R}(\mathbf{B}_9)$, broken lines that of $\mathcal{R}(\mathbf{B}_6)$ and $\mathcal{R}(\mathbf{B}_{10})$, dotted lines that of $\mathcal{R}(\mathbf{B}_7)$ and $\mathcal{R}(\mathbf{B}_{11})$, and dotted-broken lines that of $\mathcal{R}(\mathbf{B}_8)$ and $\mathcal{R}(\mathbf{B}_{12})$. Here, differently from Fig. 2, one symbol B_m is centered inside the corresponding region.

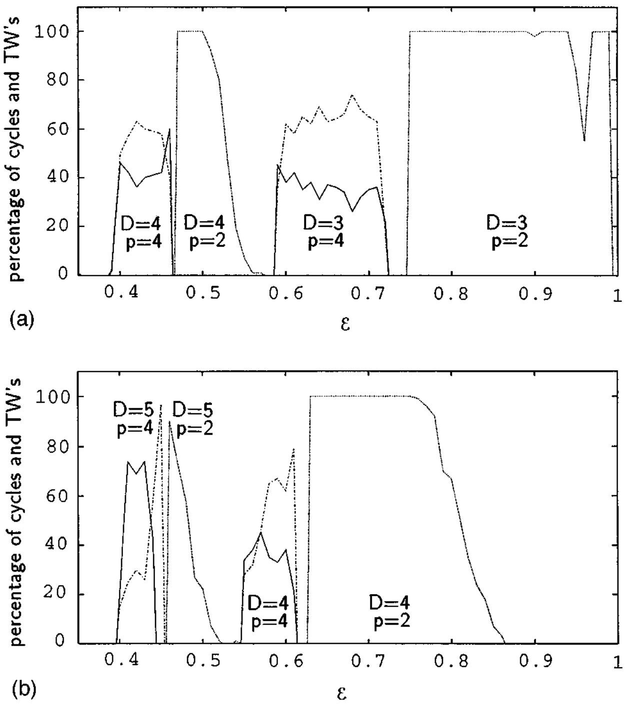

FIG. 4. $N=27$, $a=1.7$ (a); $a=2$ (b). Number of cycles of period 2 (dotted lines), cycles of period 4 (broken lines), and quasiperiodic traveling waves (continuous lines) as functions of ϵ for each allowed D, which is shown near the corresponding curves. 5×10^5 initial transients were discarded for $\epsilon\leqslant0.6$, and 3×10^5 for larger ϵ. TW's in the vertical axis label means traveling waves.

$\alpha_m^1<\alpha_{m+1}^1$ and $\alpha_m^2<\alpha_{m+1}^2$. Figure 3 clearly shows that perfectly similar relations hold for the β_m^1's and the β_m^2's as well.

IV. STABILITY STATEMENTS

Our major findings can be formalized in two *stability statements*: one for period 2, the other for period 4. They are based on the conjecture that any nonchaotic attractor of $\Phi_{\epsilon,a}^N$ with D domains, originates, in some way, from a cycle of $\Phi_{0,a}^N$ with the same spatial structure, and which is composed of $r+s=D$ elementary waves, r of length m and s of length $m+1$. By identifying the component waves with the cycles $\mathbf{A}_m$ or $\mathbf{B}_m$ depending on the period, we find that the stability of the entire structure is determined by that of the components. Note that, given N and D, one tern of integers (m,r,s) exists such that $rm+s(m+1)=N$, $r+s=D$, $r>0$, $s\geqslant0$ [i.e., $m=[N/D]$, $s=N-mD$, $r=D-s$].

The stability statements are formulated with reference to the parameter region $\mathcal{P}\equiv\{0.4\leqslant\epsilon\leqslant0.9,\ 1.6\leqslant a\leqslant2\}$. The parameter ϵ is assumed $\leqslant0.9$ because for larger values cycles may take place whose numbers of domains do not seem to follow any rule.

The stability statement for period 2 is the following: Let $\Sigma_{N,D}^{\alpha}$ be the set of all the cycles of $\Phi_{\epsilon,\overline{a}}^N$ of period 2 with D domains that are stable in the region $\mathcal{P}$. Denote by (α_D^1,α_D^2) the minimum interval of ϵ that contains all the stability ranges of the cycles of $\Sigma_{N,D}^{\alpha}$. Consider the tern (m,r,s) defined as above by N and D. Then (α_D^1,α_D^2) is included in (α_m^1,α_m^2) if $s=0$, in $(\alpha_m^1,\alpha_{m+1}^2)$ if $s>0$. Furthermore,

$$\alpha_D^1\approx\alpha_m^1+\frac{s}{D}\,(\alpha_{m+1}^1-\alpha_m^1). \tag{5}$$

An analogous statement, though a little weaker, holds for period 4: Let $\Sigma_{N,D}^{\beta}$ be the set of all the cycles and quasiperiodic traveling waves of $\Phi_{\epsilon,\overline{a}}^N$ of period 4 with D domains that are stable in $\mathcal{P}$. Denote by (β_D^1,β_D^2) the minimum interval of ϵ that contains all the stability ranges of the attractors of $\Sigma_{N,D}^{\beta}$. Then (β_D^1,β_D^2) is entirely or partially included in (β_m^1,β_m^2), with $\beta_D^1>\beta_m^1$, if $s=0$, while it is entirely included in $(\beta_m^1,\beta_{m+1}^2)$ if $s>0$. Moreover, if $\Sigma_{N,D-1}^{\beta}$ is not empty, the interval (β_D^1,β_D^2) precedes $(\beta_{D-1}^2,\beta_{D-1}^2)$, which is to say that $\beta_D^1<\beta_{D-1}^1$ and $\beta_D^2<\beta_{D-1}^2$.

We have ample data to support these statements. In order to produce our data we chose several pairs $(N,\overline{a})$, with N from 15 up to 400, and for each pair we used a grid for $\epsilon\in[0.3,1]$ with $\Delta\epsilon=0.01$. For each point $(\epsilon,\overline{a})$ we generated 100 trajectories, starting from randomly chosen initial points, and determined the corresponding attractors. In particular, we selected the cycles of periods 2 and 4, and the traveling waves of period 4. The data relative to each $(N,\overline{a})$ were collected and visualized in a picture showing, as a function of ϵ and for each D, the number of the cycles of period 2, that of the cycles of period 4, and that of the traveling waves. We point out that, owing to the fact that, in counting the attractors, only mistakes of underestimation were possible, the resultant graphs are certainly qualitatively correct. Quantitatively speaking, they are substantially correct, as is in most cases evident if one adds up the number of the cycles

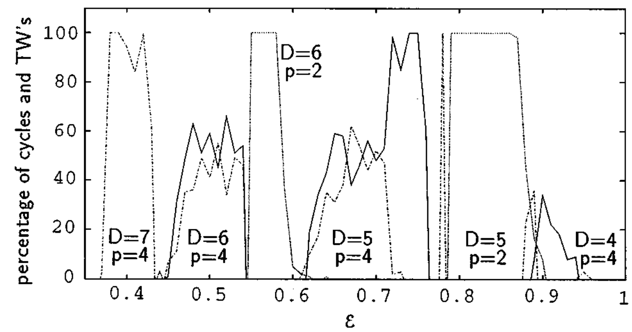

FIG. 5. $N=42$, $a=1.8$. As in Fig. 4.

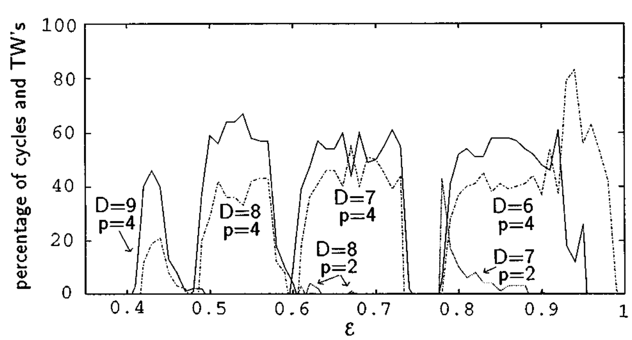

FIG. 6. $N=64$, $a=1.7$. As in Fig. 4.

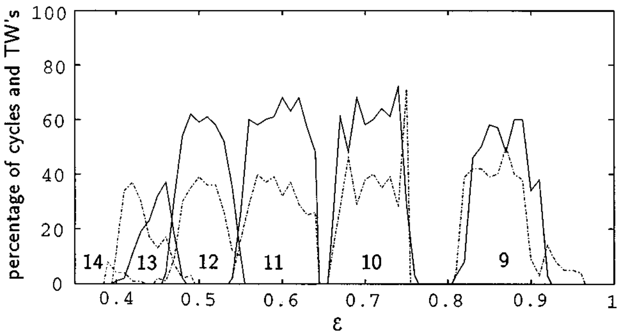

FIG. 7. $N=100$, $a=1.6$. As in Fig. 4.

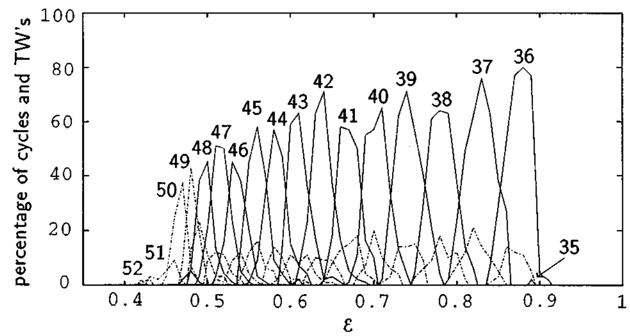

FIG. 9. $N=400$, $a=1.6$. As in Fig. 8.

and that of the traveling waves, and compares the result with the number of initial conditions.

Some of these pictures, corresponding to $N=27$, 42, 64, 100, 200, and 400, respectively, are shown in Figs. 4–9. When compared with Figs. 2 and 3, they provide strong evidence of the validity of the statements.

The stability statements settle some important points: (a) Only a few wavelengths m, $5\leqslant m\leqslant 12$, are allowed. (b) A wavelength m can occur singly or coupled with $m+1$; while it occurs singly only if $N=mD$, it occurs coupled whenever $N=rm+s(m+1)$, with $r+s=D$. (c) A given m or a given pair $(m,m+1)$ cannot occur everywhere, but only in a (sufficiently) well-defined parameter region, which depends on the tern (m,r,s) and on the period. (d) Points (a) and (c) are the consequence of a strict connection with the stability of some simple cycles whose sizes correspond exactly to the allowed wavelengths.

Not only do these results explain some previously observed facts [2], they also contain a very important element: the possibility of predicting, although approximately, the parameter region where a cycle or a quasiperiodic traveling wave with a given spatial structure may be attracting. For fixed a, the stability intervals (α_D^1,α_D^2), as well as the intervals (β_D^1,β_D^2), can be defined more or less accurately, depending on the case, by means of the intervals (α_m^1,α_m^2) and (β_m^1,β_m^2), which are known from Figs. 2 and 3. In fact, concerning period 2, each α_D^1 divides the interval $(\alpha_m^1,\alpha_{m+1}^1)$ into parts that are inversely proportional, with a fair degree of accuracy, to r and s. The result for period 4 is somewhat more approximate: the inverse proportionality of the parts in which β_D^1 divides $(\beta_m^1,\beta_{m+1}^1)$ is only very rough. In any case, the intervals (α_D^1,α_D^2), as well as the intervals (β_D^1,β_D^2), follow one another on the ϵ axis with decreasing D. Furthermore, in general, α_D^1 comes slightly after β_D^2.

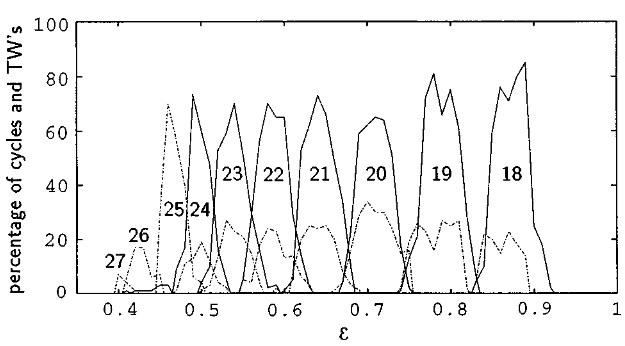

FIG. 8. $N=200$, $a=1.6$. As in Fig. 4, but with 10^6 discarded initial transients for $\epsilon\leqslant 0.6$, and 5×10^5 for larger ϵ.

The sequence of Figs. 4–9 provides a fairly good description of the nonchaotic behavior of lattice (1) in the parameter region under consideration. It shows, in particular, the phenomenon of ''pattern selection with suppression of chaos'' [2,6]. In this regard, let us stress the points that appear of most interest. Cycles and traveling waves of period 4 are by far the most important attractors. Their role in the dynamics is already rather significant for small N, but it becomes more and more relevant as N increases. In contrast, the importance of the cycles of period 2 diminishes rapidly, practically vanishing as N approaches 100. However, if one considers lattices of small size, when very few numbers D are allowed and there are wide intervals of ϵ in which neither cycles nor traveling waves of period 4 are present, the prevalent dynamics is ruled by cycles of period 2 lying on a heteroclinic cycle.

Our pictures render an account of the behavior of most of the trajectories we considered. It must be said that the remaining trajectories approach different attractors, including heteroclinic cycles and traveling waves of period 8 or larger, normal two- and three-tori, and chaotic attractors.

V. FORMATION OF STABLE STRUCTURES

In order to provide a heuristic justification for the stability statements, we made a number of numerical experiments aimed at explaining the mechanisms of formation of stable cycles and traveling waves. More precisely, we tried to confirm the following two hypotheses: (a) any attractor of $\Sigma_{N,D}^{\alpha}$ $(\Sigma_{N,D}^{\beta})$ is connected, through a more or less complicated sequence of bifurcations, with some cycle of $\Phi_{0,\bar{a}}^{N}$ of period 2 (period 4); and (b) this cycle can be suitably decomposed into D simpler structures corresponding to D cycles $\mathcal{A}_m$ $(\mathcal{B}_m)$ if $N=mD$, to r cycles $\mathcal{A}_m$ $(\mathcal{B}_m)$ and $s=D-r$ cycles $\mathcal{A}_{m+1}$ $(\mathcal{B}_{m+1})$ if $N=rm+s(m+1)$. Let us denote such a cycle by $\mathcal{A}_{N,D}^{r,s}$ $(\mathcal{B}_{N,D}^{r,s})$.

In the case of cycles of period 2, both hypotheses seem to hold for almost all choices of initial conditions and parameters. In fact, in all but one of the 15 cases we examined, we were able to connect the considered cycle of $\Sigma_{N,D}^{\alpha}$ with a cycle $\mathcal{A}_{N,D}^{r,s}$.

Regarding cycles of period 4, things are more complicated and cannot be defined as in the case of period 2. For parameter values that are not too large (roughly for $a\leqslant 1.8$

and $\epsilon_D^1 \leqslant 0.5$), there is numerical evidence that almost all cycles of $\Sigma_{N,D}^{\beta}$ can be continued backwards up to $\epsilon = 0$, where they assume the form of a $\mathcal{B}_{N,D}^{r,s}$. For larger parameter values, it is practically impossible in most cases to follow the cycle up to $\epsilon = 0$. When the attempt is successful, the structure of the cycle at $\epsilon = 0$ turns out to be more complex than that of a $\mathcal{B}_{N,D}^{r,s}$. In any case, this can again be seen as a combination of D simpler structures corresponding to cycles of form (3) which are only in part identifiable with cycles $\mathcal{B}_m$ or $\mathcal{B}_{m+1}$.

The possibility of connecting a traveling wave with a cycle $\mathcal{B}_{N,D}^{r,s}$ is subordinate to the possibility of explaining the appearance of the traveling wave through a bifurcation from some stable cycle of $\Phi_{\epsilon,\bar{a}}^N$. In this regard, our experiments show that a mechanism sometimes occurs which fulfills the former possibility.

The mechanism is as follows: As ϵ increases from zero, the continuation of a $\mathcal{B}_{N,D}^{r,s}$ acquires stability, and then it becomes involved as a node in a phase-locked torus with a rotation number $w = 0$. The phase-locked torus is formed by the unstable manifold of a saddle that joins the node. The torus is not ''visible'' because the attractor is the node. If ϵ increases again, the node may disappear by collapsing with the saddle. When this happens, it makes the torus appear with w very close to zero. Because w remains very small as ϵ is increased further, a traveling wave has occurred.

Such a mechanism for the formation of a traveling wave was carefully checked in a few cases, for instance, for $N = 45$, $D = 6$, and $a = 1.60$ and $N = 64$, $D = 8$, and $a = 1.69$. It must be said, however, that in the great majority of cases the appearance of a traveling wave is not directly related to the disappearance (or the loss of stability) of a stable cycle. In our opinion, a connection with a cycle of $\Phi_{0,\bar{a}}^N$, perhaps never stable, always exists, but it is numerically undetectable.

VI. CONCLUSION

The results of this paper concern the behavior of a coupled map lattice (1) for high nonlinearity and medium and large coupling. For such parameter values the dynamics is largely governed by cycles and traveling waves of period 4 and, in lattices of small size N, by cycles of period 2 as well. We have investigated these attractors and have summarized our findings in two stability statements.

In particular, we have found that, regardless of N, the stability of a spatial structure (cycle or traveling wave) is associated with that of a few basic cycles $\mathcal{A}_m$ or $\mathcal{B}_m$ of small size m. Such an association is based on the decomposition of the structure in terms of elementary waves of size m and $m+1$. By identifying these waves with the $\mathcal{A}_m$'s or $\mathcal{B}_m$'s, it is possible, thanks to the fact that their stability regions are (numerically) known, to deduce approximately where a cycle or traveling wave with a given spatial structure may be attracting.

In addition to this, we have demonstrated that almost all the cycles of period 2 and also, for suitable parameter values, most of the cycles of period 4, originate from cycles of the uncoupled lattice, which are composed precisely of r blocks $\mathcal{A}_m$ (or $\mathcal{B}_m$) and s blocks $\mathcal{A}_{m+1}$ (or $\mathcal{B}_{m+1}$). Furthermore, in some cases a traveling wave is also directly connected with a similarly structured cycle of $\Phi_{0,\bar{a}}^N$. This connection of both cycles and traveling waves with the cycles of the lattice at $\epsilon = 0$ and with the basic cycles $\mathcal{A}_m$ and $\mathcal{B}_m$ seems very important. On the one hand, in fact, it provides a justification for the stability statements; on the other hand, it suggests a possible starting point for a theoretical approach aimed at rigorous results.

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